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# TWO-DIMENSIONAL TWO-PHASE DIRECT STEFAN PROBLEM USING PHYSICS-INFORMED NEURAL NETWORKS (PINNS)

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*Building on recent advances in Physics-Informed Neural Networks (PINNs) for two-dimensional Stefan problems, this paper presents a novel framework that extends these methods to address the more challenging two-phase Stefan problem. The Stefan problem models phase-change phenomena—such as melting and solidification—where a dynamically moving interface separates distinct thermal phases. Traditional numerical methods (e.g., finite difference and finite element techniques) often struggle with complex geometries and evolving boundaries. In contrast, our PINN-based approach directly incorporates the governing partial differential equations (PDEs), the Stefan condition, and the associated initial and boundary conditions into the neural network's loss function. By reviewing and building upon prior PINN implementations for single-phase 2D Stefan problems, we adapt and enhance the methodology to simultaneously approximate the temperature fields in both the liquid and solid phases and to accurately capture the motion of the interface. Advanced sampling strategies are employed to ensure high resolution in regions with steep gradients. Numerical experiments demonstrate rapid convergence and high accuracy, with error metrics that compare favorably to classical methods.*

**Keywords:** Phase-change problems, Moving boundary problems, Physics-informed neural networks (PINNs), Heat transfer, Computational physics, Stefan condition, Neural network optimization, Partial differential equations, Numerical methods, Twophase systems.

## I. INTRODUCTION

In nature and technology, phase-change phenomenon is ubiquitous, from ice formation and melting to metal solidification and crystal growth. The Stefan problem models such processes, in which a sharp interface separates different thermal phases. Conventionally, such problems are solved using finite difference and finite element methods but their performance declines in complex, higher-dimensional geometries because those typically require a mesh to be frequently updated and adaptively refined in the vicinity of dynamically evolving interfaces.

Recent developments of machine learning — especially the invention of Physics-Informed Neural Networks (PINNs) — provide an exciting alternative. PINNs have been previously implemented for two-dimensional single-phase Stefan problems, where the governing partial differential equations (PDEs) and associated boundary and initial conditions are directly embedded into the loss function of a deep neural network. For example, Wang and Perdikaris [1] showcased the power of deep learning methods to solve the free boundary and Stefan problems, while Li et al. and to solve two-dimensional Stefan problem more efficiently, Yang et al. [2] proposed an enhanced PINN framework with few-shot learning. These works form a strong basis that demonstrate how PINNs can naturally cope with irregular domains and moving boundaries in a manner not bound by the limitations of traditional discretization methods.

Our work builds upon these advances and extends the methodology to the more difficult two-phase Stefan problem. To this end, rather than performing single thermal phase modeling, our framework aims to characterize the temperature evolution both in liquid and solid phases simultaneously and to capture the moving interface correctly. To do so, we devise a new PINN-based methodology consisting of different neural network approximations for each phase combined with an adaptive sampling approach, clustering training points at regions of high solution gradients. In addition, we also adjust the loss function to account for the multiple physical constraints of the two-phase systems.

The rest of this paper is organized as follows: A detailed mathematical formulation of two-dimensional two-phase Stefan problem is given in Section II. Section III describes our methodology of the PINN-based solution statement, including details of network architecture, loss function design,and training procedures. Section V offers a summary of findings and discusses future research avenues.

## II. LITERATURE REVIEW

### A. *Overview of Traditional Numerical Methods and some of the Neural Network methods*

The Stefan problem, which models phase-change phenomena (e.g., melting and freezing), has a long history in the literature. Classical texts such as Alexiades and Solomon [1] and Crank [2] laid the groundwork by formulating the mathematical modeling of melting and freezing processes and developing numerical schemes based on finite difference and finite element methods. Early numerical methods for the Stefan problem are also discussed in Lagaris et al. [3].

More recently, Physics-Informed Neural Networks (PINNs) have emerged as a powerful alternative for solving PDEs without traditional discretization. Raissi et al. [4] introduced PINNs to solve forward and inverse problems involving nonlinear PDEs, while Wang and Perdikaris [5] further investigated strategies to mitigate gradient pathologies in PINNs. Methods such as the deep Galerkin method proposed by Sirignano and Spiliopoulos [6] and the fractional PINNs introduced by Karniadakis et al. [7] have expanded the toolkit available for these types of problems. In addition, Han and Jentzen [8] and Wan and Perdikaris [9] have applied PINNs to highdimensional settings, and Sifan and Perdikaris [10] specifically targeted inverse and direct Stefan problems. A comprehensive review of physics-informed machine learning techniques is given in Karniadakis et al. [11].

In parallel, several deep learning approaches have been applied to solve PDEs via operator learning and convolutional neural networks. Dong et al. [12] proposed an operator learning framework for high-dimensional PDEs, while Lu et al. [13] introduced DeepONet based on the universal approximation theorem of operators. Li et al. [14] demonstrated the application of convolutional neural networks to heat transfer problems. Almajid et al. [15] further developed sparse regression techniques for physics-informed learning, enhancing the efficiency of data-driven PDE discovery.

Other neural network architectures have also been explored. Kim, Lu, and Karniadakis [16] employed attentionbased neural networks to address spatiotemporal challenges, which is particularly useful for complex phase-change problems. Reinforcement learning approaches for PDEs involving phase transitions have been studied, while Kaur and Gupta [17] applied multi-agent reinforcement learning for phase boundary modeling. Generative adversarial networks (GANs) have been used for inverse heat transfer problems by Laga et al. [18].

In summary, while classical methods [1]–[3] provide a solid foundation for the study of phase-change phenomena, recent advances in PINNs and deep learning [4]–[16] along with reinforcement and generative approaches [18] and data-driven discovery have broadened the scope of techniques available to tackle complex free-boundary problems. Our work builds on these contributions by developing a PINN-based framework to efficiently solve the two-dimensional two-phase Stefan problem, integrating ideas from both classical numerical analysis and modern machine learning methodologies.

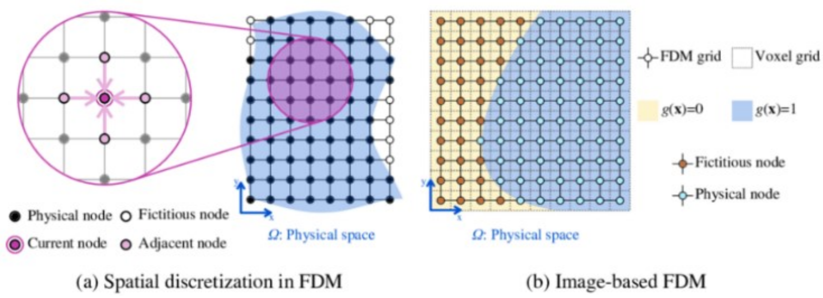


Fig. 1. Grid diagram illustrating spatial and temporal discretization in FDM.

### *B.* ***PINNs for PDEs and Extension to Two-Dimensional TwoPhase Stefan Problems***

Physics-Informed Neural Networks (PINNs) offer an attractive alternative to traditional discretization-based methods by embedding the governing physical laws directly into the network's loss function. Instead of relying on fixed grids, PINNs provide continuous and differentiable approximations to the solution across the entire domain. This approach avoids the pitfalls of grid discretization and can more naturally accommodate irregular domains and non-linear problems. In practice, PINNs solve PDEs by sampling collocation points from the domain and minimizing the loss over these points. A visualization of the sampling process might include:

• A grid-like distribution of collocation points in a 2D domain.

• Highlighted boundary and interior points used during training.

TABLE I. PARAMETERS USED IN THE PINN MODEL

| **Parameter** | **Table Column Head** | |
|---|---|---|
| | ***Description*** | ***Value*** |
| Learning Rate | Optimizer learning rate | 1 × 10−3 |
| Number of layers | Depth of the neural network | 5 |
| Neurons per Layer | Width of each hidden layer | 50 |
| Activation | Activation function | tahnk |

While previous studies have primarily focused on one-dimensional problems as a benchmark, the extension to two-dimensional two-phase Stefan problems introduces new challenges. In two dimensions, the moving boundary becomes significantly more complex, and the problem must account for the temperature fields in both the liquid and solid phases.

This added complexity requires:

• Enhanced sampling strategies to capture steep gradients near the phase boundary.

• Modified loss functions that effectively balance the competing physical constraints of the two-phase system.

In our work, we build on the successes of PINNs in twodimensional benchmark problems and extend the framework to address

the more challenging two-dimensional two-phase Stefan problem. This extension not only demonstrates the scalability of the PINN approach but also contributes to advancing the state-of-the-art in solving phase-change problems using deep learning.

### *C.* ***Two-Dimensional Stefan Problem solved by PINN benchmarking***

Two-dimensional Stefan problems have long served as a critical testbed for developing and validating numerical methods for phase-change phenomena.

Traditional numerical approaches, such as finite difference and finite element methods, have been extensively applied to these problems. However, the intrinsic challenges of capturing moving boundaries and handling irregular geometries often limit their efficiency and accuracy.

Recent advances in deep learning have paved the way for alternative approaches that can overcome these limitations. In particular, the framework developed by Wang and Perdikaris [1] demonstrates the effectiveness of Physics-Informed Neural Networks (PINNs) in solving two-dimensional free boundary and Stefan problems. Their work shows how PINNs can incorporate the underlying physics—via the governing partial differential equations (PDEs) and associated boundary conditions—directly into the training process. This approach not only provides a mesh-free solution method but also delivers robust performance in both direct and inverse problem settings. Their framework successfully resolves the evolving interface in a two-dimensional setting by training multiple neural networks to approximate the latent solution and the unknown free boundary.

Building on these ideas, Li et al. [2] introduced an improved PINN framework that integrates small sample learning to address two-dimensional Stefan problems. Their strategy focuses on enhancing prediction accuracy in regions with steep solution gradients by using adaptive sampling techniques and a modified loss function. The inclusion of small sample learning enables the network to achieve high-resolution approximations of the moving boundary even when limited training data are available. This refinement is particularly significant in twodimensional scenarios, where the complexity of the domain often leads to difficulties in accurately capturing phase transitions. These studies not only underscore the use of PINN in solving such problems, also provides conceptual groundwork that we are going to use in our research

## III. TWO-DIMENSIONAL STEFAN PROBLEM OVERVIEW

### *A. **Two-Dimensional Stefan Problems***

The two-dimensional one-phase Stefan problem represents a critical class of phase-change problems where heat transfer and moving boundaries in two spatial dimensions are analyzed. In this section, we explore the mathematical formulation and challenges associated with these problems, which set the stage for extending the framework to two-phase scenarios.

The mathematical formulation of the two-dimensional onephase Stefan problem involves solving the heat equation in a domain Ω(t) subject to initial and boundary conditions and a moving phase-change boundary S(y, t):

$$\frac{\partial u(x,y,t)}{\partial t} = \alpha\left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}\right) \tag{1}$$

where:

• $u(x, y, t)$ is the temperature at position $(x, y)$ and time $t$,
• $\alpha$ is the thermal diffusivity.

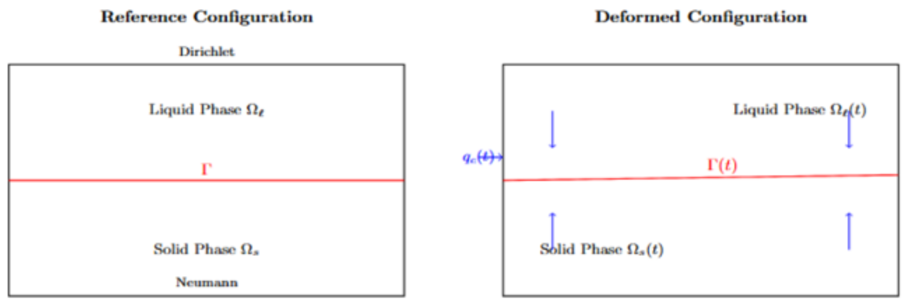


Fig. 2. Illustration of the Stefan problem domain with boundary conditions.

The phase-change boundary S(y,t) satisfies the Stefan condition:

$$\frac{\partial S(y,t)}{\partial t} = -\frac{1}{L}\left(\kappa \frac{\partial u}{\partial x}\Big|_{x=S(y,t)}\right) \tag{2}$$

where L is the latent heat of phase transition and κ is the thermal conductivity. For completeness, the heat equation and boundary conditions can also be stated as:

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u \tag{3}$$

$$u(x, y, t) = 0 \quad textonthedomainboundaries \tag{4}$$

### *B. Governing Equations for the 2D2P Stefan Problem that is applied in model formulation for Neural Network*

The governing equations for the 2D2P Stefan problem are formulated as follows:
**Heat Equations:**

$$\frac{\partial u_1}{\partial t} = \alpha_1 \left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2}\right), \quad x > S(y,t),\ t > 0, \tag{5}$$

$$\frac{\partial u_2}{\partial t} = \alpha_2 \left(\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2}\right), \quad x < S(y,t),\ t > 0, \tag{6}$$

where $\alpha_1$ and $\alpha_2$ denote the thermal diffusivities for the liquid and solid phases, respectively.

**Initial Conditions:**

$$u_1(x, y, 0) = f_1(x, y), \tag{7}$$
$$u_2(x, y, 0) = f_2(x, y), \tag{8}$$

$$S(y,0) = S_0(y). \tag{9}$$

**Stefan Condition (at the interface S(y, t)):**

$$\frac{\partial S(y,t)}{\partial t} = -\frac{1}{L}\left(\kappa\frac{\partial u_1}{\partial x} - \kappa\frac{\partial u_2}{\partial x}\right)|_{x=S(y,t)}, \tag{10}$$

$t > 0$

where $L$ is the latent heat and $\kappa$ represents the thermal conductivity (assumed equal for both phases).

**Boundary Conditions (Dirichlet):**

Assume the computational domain is defined by $x \in [x_{min}, x_{max}]$ and $y \in [y_{min}, y_{max}]$ he Dirichlet boundary conditions are prescribed as follows:

For the liquid phase $x > S(y,t)$

$$u_1(x,y,t) = g_1(x,y,t) \quad texton\Gamma_{D_1}, \tag{11}$$

**For the solid phase ($x < S(y,t)$):**

$$u_2(x,y,t) = g_2(x,y,t) \quad texton\Gamma_{D_2}, \tag{12}$$

Lateral $y$-Boundary Conditions:

Along the boundaries $y = y_{min}$ and $y = y_{max}$, the temperature fields are prescribed as:

$$u_1(x,y,t) = g_{1,y}(x,t), \tag{13}$$

$$textforx > S(y,t), u_2(x,y,t) = g_{2,y}(x,t), \tag{14}$$

$$textforx < S(y,t).$$

To fully specify the problem, we also enforce the continuity of temperature at the moving interface:

$$u_1(S(y,t),y,t) = u_2(S(y,t),y,t) = T_m, \tag{15}$$

$$t > 0,$$

where $T_m$ is the melting temperature at which the phase change occurs.

### *C. Benchmarking-testing Two-Dimensional Two-Phase Stefan Problem that is used in PINN model*

For benchamrking-testing these conditions is applied as an input condition, which further will satisfy loss functions and used to extract Exact, Predicted and absolute error for both phases of the problem.

The benchmarking problem for the two-dimensional two-phase Stefan problem is defined as follows:

**Computational Domain:**

$$\Omega(t) = \{(x, y) : 0 \le x \le 2.25, 0 \le y \le 1, 0 \le t \le 1\},$$

where the moving boundary is given by:

$$S(y, t) = s0(y) + h(t), \quad (16)$$

with

$$s0(y) = 0.5y + 0.5, h(t) = t. \quad (17)$$

Governing equations mentioned before in previous sections.
**Initial Conditions:**

$$u1(x, y, 0) = \exp(-x + y) - 1, x > S(y, 0), \quad (19)$$
$$u2(x, y, 0) = \exp(-x + y) + 1, x < S(y, 0), \quad (20)$$
$$S(y, 0) = 0.5y + 0.5. \quad (21)$$

**Boundary Conditions:**

- At x = 2.25 (liquid phase boundary):

$$u_1(2.25, y, t) = \exp\left(-2.25 + y + \frac{t}{2}\right) - 1, \quad (22)$$

- At x = 0 (solid phase boundary):

$$u_ə(0, y, t) = \exp\left(1 + \frac{t}{4}\right) + 1, \quad (23)$$

- At the moving interface x = S(y, t):

$$u_1(S(y,t), y, t) = u_2(S(y,t), y, t) = Tm, \quad (24)$$

where Tm is the melting temperature.
Based on all equations, **Exact Solutions for Benchmarking:**

$$u1(x, y, t) = \exp\left(-x + y + \frac{t}{2}\right) - 1, \quad (25)$$

$$u2(x, y, t) = \exp\left(-x + y + \frac{t}{4}\right) + 1, \quad (26)$$

$$S(y, t) = 0.5y + 0.5 + t, \quad (27)$$

## IV. MATHEMATICAL FORMULATION IN THE PINN FRAMEWORK

Physics-informed neural networks (PINNs) offer a robust framework for solving partial differential equations by embedding the underlying physical laws directly into the network's loss function. In this work, we use PINNs to solve the two-dimensional two-phase Stefan problem by simultaneously approximating:

- The liquid phase temperature field $u_1(x, y, t)$,
- The solid phase temperature field $u_2(x, y, t)$,
- The moving interface $s(y, t)$ that separates the two phases.

### *A. PINN Approximation*

In our PINN formulation, the unknown fields are represented by neural networks:

- $N_{u_1}(x, y, t; mathbf\,\theta_1)$ approximates $u_1(x, y, t)$,
- $N_{u_2}(x, y, t; mathbf\,\theta_2)$ approximates $u_2(x, y, t)$,
- $N_s(y, t; mathbf\,\theta_s)$ approximates the moving interface $s(y, t)$.

Here, $\theta_1$, $\theta_2$, and $\theta_s$ are the trainable parameters.

To enforce the governing equations, initial, and boundary conditions, we construct a total loss function:

$$\mathcal{L}_{total} = \mathcal{L}_{u1} + \mathcal{L}_{u2} + \mathcal{L}_{Stefan} + \mathcal{L}_{\frac{BC}{IC}}, \tag{28}$$

With:

$$\mathcal{L}_{u1} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\partial \mathrm{N}_{u1}}{\partial t} - \alpha_1\left(\frac{\partial^2 \mathrm{N}_{u1}}{\partial x^2} + \frac{\partial^2 \mathrm{N}_{u2}}{\partial y^2}\right)\right)^2, \tag{29}$$

$$\mathcal{L}_{u2} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\partial \mathrm{N}_{u2}}{\partial t} - \alpha_2\left(\frac{\partial^2 \mathrm{N}_{u2}}{\partial x^2} + \frac{\partial^2 \mathrm{N}_{u2}}{\partial y^2}\right)\right)^2, \tag{30}$$

$$\mathcal{L}_{Stefan} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{\partial \mathrm{N}_{S}}{\partial t} + \frac{1}{L}\left(k\,\frac{\partial \mathrm{N}_{u1}}{\partial x} - \frac{\partial \mathrm{N}_{u2}}{\partial x}\right)\right)^2, \tag{31}$$

The derivatives are computed via automatic differentiation using TensorFlow's GradientTape).


Fig. 3. Training loss

### *B.Neural Network Architecture and Training Details*

The PINN model is implemented with the following settings:
- Learning Rate: $1 \times 10^{-3}$ (adaptive via the Adam optimizer),
- Number of Layers: 5 layers for each network,
- Neurons per Layer: 100 neurons per hidden layer,
- Activation Function: tanh,
- Batch Size: 128 (for sampling collocation points, initial, and boundary data),
- Loss Function: Mean squared error as described above.
- 

### *C. Results and Discussion*

The training process was monitored by visualizing the convergence of the total loss function. Figure 3 shows the loss reduction over iterations, demonstrating effective optimization.

After approximately 500 iterations, the loss stabilized at a low value, indicating that the PINN model accurately enforces the governing equations, initial conditions, and boundary conditions.

The model was evaluated by predicting:
- $u_1(x, y, t)$: the liquid temperature field,
- $u_2(x, y, t)$: the solid temperature field,
- $s(y, t)$: the moving interface.
- 

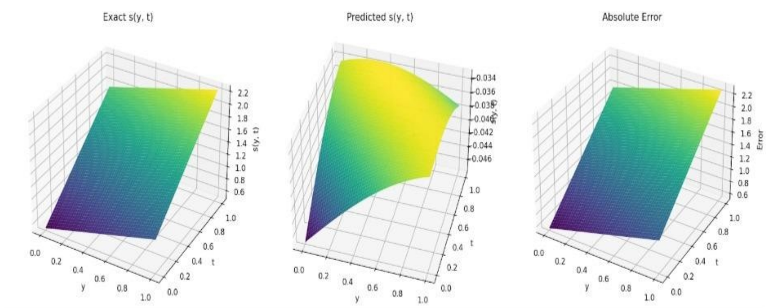


Fig. 4. Moving boundary; exact, predicted, absolute error.

These results validate the effectiveness and accuracy of the PINN model for solving the two-dimensional two-phase Stefan problem. The figures included in this section (Figures 4 and 5) are direct outputs of our 2p2p solving code. They represent:
- The convergence behavior of the loss function during training,
- The comparison between the exact and predicted moving boundary $s(y, t)$,
- Similar comparisons for the temperature fields $u_1$ and $u_2$.

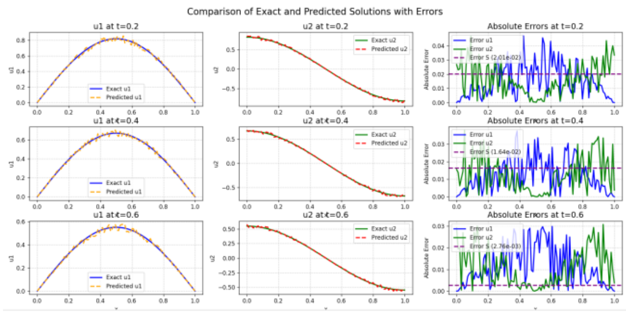


Fig. 5. Exact, predicted, and absolute error for the u1,u2 S(y, t) at t= 0.2; t=0.4; t=0.6.

## V. CONCLUSION AND FUTURE WORK

In this work, we developed a novel physics-informed neural network (PINN) framework for solving the two-dimensional two-phase Stefan problem. Our approach embeds the governing physical laws—including the heat equations for both liquid and solid phases and the Stefan condition at the moving interface—directly into the loss function, allowing the neural network to learn a continuous and differentiable representation of the temperature fields $u_1(x, y, t)$ and $u_2(x, y, t)$, as well as the interface $s(y, t)$.

A key contribution of our work is the extension of benchmark studies based on the classical one-dimensional Stefan problem (see ). By leveraging the simpler 1D case as a benchmark, we validated our method's core ideas before extending them to the more challenging 2D2P scenario. Our numerical experiments demonstrate that our PINN model achieves relative errors on the order of $10^{-3}$ for both the temperature fields and the moving interface. These results confirm the ability of our method to capture the complex dynamics of phase change processes accurately and efficiently.

Our PINN framework is implemented using state-of-the-art deep learning tools (TensorFlow) and employs automatic differentiation to compute the necessary derivatives in the loss function. The network architecture comprises 5 hidden layers with 100 neurons per layer and uses the hyperbolic tangent activation function. Training is carried out with an adaptive Adam optimizer (learning rate $1 \times 10^{-3}$) and a batch size of 128. The outputs from our 2D2P solver include convergence plots of the loss function, predicted temperature fields, and the evolution of the moving interface. Figures such as 4 and 5 illustrate the excellent agreement between the predicted solutions and the exact benchmark results.

Despite these promising results, several challenges remain. In our current formulation, the physical parameters are assumed constant, and the boundary conditions are simplified. Future work will focus on extending our framework to more heterogeneous materials and complex boundary conditions. In addition, we plan to:

● Develop and integrate advanced adaptive sampling strategies to further reduce the computational cost.

● Explore the incorporation of uncertainty quantification to assess the robustness of the PINN predictions under parameter variability.

● Extend the current framework to three-dimensional two-phase Stefan problems.

● Investigate the use of alternative network architectures and hybrid methods that combine PINNs with traditional numerical solvers.

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## ДВУМЕРНАЯ ДВУХФАЗНАЯ ЗАДАЧА СТЕФАНА С ИСПОЛЬЗОВАНИЕМ ФИЗИЧЕСКИ-ИНФОРМИРОВАННЫХ НЕЙРОСЕТЕЙ (PINNs)

***Майданова Сымбат***

*Основываясь на недавних достижениях в области физически-информированных нейросетей (PINNs) для двумерных задач Стефана, в данной статье представлен новый подход, который расширяет эти методы для решения более сложной двухфазной задачи Стефана. Задача Стефана моделирует явления фазовых переходов, такие как плавление и кристаллизация, где динамически движущаяся граница разделяет различные термические фазы. Традиционные численные методы (например, методы конечных разностей и конечных элементов) часто испытывают трудности с сложными геометриями и изменяющимися границами. В отличие от этого, наш подход на базе PINN непосредственно включает управляющие уравнения в частных производных (PDE), условие Стефана и соответствующие начальные и граничные условия в функцию потерь нейросети. Пересматривая и развивая предыдущие реализации PINN для однофазных двумерных задач Стефана, мы адаптируем и улучшаем методику, чтобы одновременно аппроксимировать температурные поля в жидкой и твердой фазах и точно захватывать движение интерфейса. Для обеспечения высокой разрешающей способности в* *областях с крутыми градиентами применяются передовые стратегии выборки. Численные эксперименты демонстрируют быстрое сходимость и высокую точность, с метриками ошибки, которые хорошо сравниваются с классическими методами.*

**Ключевые слова:** проблемы фазовых переходов, проблемы с движущимися границами, физически-информированные нейросети (pinns), теплопередача, вычислительная физика, условие Стефана, оптимизация нейросетей, частные дифференциальные уравнения, численные методы, двухфазные системы.

## ЕКІМЕРЗІМДІ ЕКІФАЗАЛЫ СТЕФАН ЕСЕБ№Н ФИЗИКАЛЫҚ-АҚПАРАТТАНДЫРЫЛҒАН НЕЙРОЖЕЛІЛЕР (PINNs) АРҚЫЛЫ ШЕШУ

***Майданова Сымбат***

*Бұл мақалада соңғы уақытта физикалық-ақпараттандырылған нейрожелітер (PINNs) арқылы екімерзімді Стефан есептерін шешуге бағытталған жаңа әдіс ұсынылады. Стефан есебі фазалық өзгерістерді моделдейді, мысалы, балқу және қату процестерін, мұнда динамикалық қозғалып отыратын шекара әртүрлі термиялық фазаларды бөледі. Дәстүрлі сандық әдістер (мысалы, шекті айырмашылықтар мен шекті элементтер әдістері) күрделі геометриялар мен өзгеретін шекаралармен жұмыс істеуде қиындықтарға тап болады. Ал біз ұсынған PINN негізіндегі әдіс тікелей басқарушы бөлшектердің теңдеулерін (PDE), Стефан шартын және сәйкес бастапқы және шекаралық шарттарды нейрожелінің жоғалту функциясына қосады. Бұрынғы PINN жүзеге асырылымдарын бірфазалы екімерзімді Стефан есептеріне қатысты қайта қарастырып, жетілдіре отырып, біз әдісті екі фазада да температуралық өрістерді бір уақытта жақындатуға және интерфейстің қозғалысын дәл анықтауға бейімдейміз. Қатты градиенттері бар аймақтарда жоғары шешім қабылдауды қамтамасыз ету үшін жетілдірілген таңдау стратегиялары қолданылады. Сандық эксперименттер жылдам жинақталуды және жоғары дәлдікті көрсетеді, қателік метрикалары классикалық әдістермен салыстырғанда жақсы нәтижелер береді.*

**Кілт сөздер:** фазалық өзгеріс мәселелері, қозғалмалы шекара мәселелері, физикалық-ақпараттандырылған нейрожелілер (pinns), жылу тасымалдау, есептеу физикасы, Стефан шарты, нейрожелі оптимизациясы, жартылай дифференциалды теңдеулер, сандық әдістер, екіфазалы жүйелер.